A relaxed characterization of ISS for periodic systems with multiple invariant sets

Denis Efimov, Johannes Schiffer, Nikita Barabanov, Romeo Ortega

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Abstract

A necessary and sufficient criterion to establish input-to-state stability (ISS) of nonlinear dynamical systems, the dynamics of which are periodic with respect to certain state variables and which possess multiple invariant solutions (equilibria, limit cycles, *etc*.), is provided. Unlike standard Lyapunov approaches, the condition is relaxed and formulated via a sign-indefinite function with sign-definite derivative, and by taking the system's periodicity explicitly into account. The new result is established by using the framework of cell structure and it complements the ISS theory of multistable dynamics for periodic systems. The efficiency of the proposed approach is illustrated via the global analysis of a nonlinear pendulum with constant persistent input.

I. Introduction

Stability of dynamical systems is one of the fundamental problems studied in control systems theory [8], [10], [20], [22], [23], [24], [29], [34] and related domains, such as mechanics, electric circuits, power systems, systems biology, *etc*. In a general (nonlinear) setting, the main approach employed for stability analysis is based on Lyapunov theory [29]. A key advantage of a Lyapunov-based stability analysis is that boundedness and convergence properties of the solutions can be assessed without explicit computation of the latter. Instead, it suffices to verify some inequalities for the Lyapunov function and its time derivative, which is derived with respect to the system's equations. More precisely, the existence of a continuously differentiable (or at least Lipschitz continuous) Lyapunov function, which is positive definite with respect to an equilibrium (or an invariant set) and the time derivative of which is negative definite along the solutions of the system under investigation, is equivalent to stability of that dynamical system with respect to the equilibrium (or the set). Similarly, instability of an equilibrium can be studied using the Chetaev function approach [10], [16]. A Chetaev function may be sign-indefinite[1] with a negative definite derivative. There are several extensions of Lyapunov theory, including ISS and related notions [38], [11] as well as uniform stability [28], all of which allow to account for robustness in the presence of external inputs.

Classical stability theory is mainly concerned with the analysis of a single equilibrium. However, in numerous applications, such as biological or power systems, there exist several equilibria or invariant sets (including hidden attractors [13]). Hence, the rigorous analysis of such systems with several disjoint invariant sets represents an important special case of stability theory, which requires suitable methods [5], [30], [20], [33], [7], [40], [18], [14]. For this case the stability notions have to be significantly modified and relaxed as, in particular, it has been done in [14] and further in [3], [4] for the ISS case. See also [2], [6], [9] for other results on robust stability analysis of multistable systems. The main result of [4] provides necessary and sufficient conditions under which a system is stable with respect to multiple invariant solutions, which belong to a decomposable set (see Definition 3 below). Then, convergence of all solutions of the system to this set is equivalent to 1) the existence of a nonnegative (taking zero value on some of that sets) Lyapunov function, which is continuously differentiable

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[1]A function $V:\mathbb{R}^n\to\mathbb{R}$ is called sign-definite if $V(0)=0$ and $V(x)>0$ for all $x\in\mathbb{R}\setminus\{0\}$ or $V(x)<0$ for all $x\in\mathbb{R}\setminus\{0\}$; and it is called sign-indefinite if $V(x)$ takes both, positive and negative, values.

on the manifold where the system dynamics evolves and 2) the time derivative of this Lyapunov function is negative definite along the solutions of the system and only vanishes at elements of the decomposable invariant set.

Despite the significant advances achieved in [3], [4] and the generality of the results therein, their application to periodic systems is, in many cases, not straightforward. The main reason for this is the technical difficulty of constructing Lyapunov functions. For example, when some of the states of the system are periodic (*e.g.* evolve on the circle), the corresponding Lyapunov function of [3], [4] has also to be periodic, which is a severe requirement. Paramount examples of such systems are the forced nonlinear pendulum [17], [19], power systems [32], [35], [43], [44], microgrids [36], [42], phase-locked loops [25], [26], and complex networks of oscillators [40], [12], [41].

Motivated by this wide range of potential applications, we consider a special class of systems, which possess periodic right-hand sides with respect to a part of the state vector. For such systems, the present paper extends the results of [3], [4] by relaxing the requirements on differentiability and positive definiteness of the Lyapunov function. To establish the result, we use the framework of *cell structure* proposed in [27] (and later in [31]) and developed in [20], [46] for autonomous systems. As in [3], [4], under the aforementioned relaxed assumptions, necessary and sufficient conditions for ISS are derived. The presented framework is tested by applying it to a nonlinear pendulum with constant permanent input.

The outline of this paper is as follows. Preliminaries and the theories from [4] and [20], [46] are given in Section II. The problem statement is given in Section III with the main results in Section IV. The efficiency of the presented robust stability conditions is illustrated by means of the example of a nonlinear pendulum in Section V.

II. Preliminaries

Following [4], for an n-dimensional $\mathcal{C}^2$ connected and orientable Riemannian manifold M without a boundary, let the map $f(x,d) : M \times \mathbb{R}^m \to T_xM$ (where T_xM is the tangent space of M at x) be of class $\mathcal{C}^1$, and consider a nonlinear system of the following form:

$$\dot{x}(t) = f(x(t), d(t)), \tag{1}$$

where the state $x(t) \in M$ and $d(t) \in \mathbb{R}^m$ (the input $d(\cdot)$ is a locally essentially bounded and measurable signal) for $t \geq 0$. We denote by $X(t, x; d)$ the uniquely defined solution of (1) at time t fulfilling $X(0, x; d) = x$. Together with (1) we will analyze its unperturbed version:

$$\dot{x}(t) = f(x(t), 0). \tag{2}$$

A set $S \subset M$ is invariant for the unperturbed system (2) if $X(t, x; 0) \in S$ for all $t \in \mathbb{R}$ and for all $x \in S$; for $x \in M$ the point $y \in M$ belongs to its ω-limit (α-limit) set if there is a sequence t_i, $\lim_{i\to+\infty} t_i = +\infty$, such that $\lim_{i\to+\infty} X(t_i, x; 0) = y$ ($\lim_{i\to+\infty} X(-t_i, x; 0) = y$); for any $x \in M$ its α- and ω-limit sets are invariant [21]. Define the distance from a point $x \in M$ to the set $S \subset M$ as $|x|_S = \inf_{a\in S} \delta(x, a)$, where the symbol $\delta(x_1, x_2)$ denotes the Riemannian distance between x_1 and x_2 in M, $|x| = |x|_{\{0\}}$ for $x \in M$ (0 is a point selected on M) or a usual Euclidean norm of a vector $x \in \mathbb{R}^n$. For a signal $d : \mathbb{R} \to \mathbb{R}^m$ the essential supremum norm is defined as $\|d\|_\infty = \operatorname{ess\,sup}_{t\geq 0} |d(t)|$.

A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if $\alpha(0) = 0$ and the function is strictly increasing. The function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is increasing to infinity. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}_\infty$ for each fixed $t \in \mathbb{R}_+$ and $\lim_{t\to+\infty} \beta(s, t) = 0$ for each fixed $s \in \mathbb{R}_+$.

The notation $DV(x)f(x)$ stands for the directional (or Dini) derivative of a continuously differentiable (or locally Lipschitz continuous) function $V : \mathbb{R}^n \to \mathbb{R}_+$ with respect to the vector field f evaluated at the point x.

A. Decomposable sets

Let $\Lambda \subset M$ be a compact invariant set for (2).

Definition 1. [30] A decomposition of Λ is a finite and disjoint family of compact invariant sets $\Lambda_1, \ldots, \Lambda_k$ such that

$$\Lambda = \bigcup_{i=1}^{k} \Lambda_i.$$

For an invariant set Λ, its attracting and repulsing subsets are defined as follows:

$$\mathfrak{A}(\Lambda) = \{x \in M : |X(t,x;0)|_{\Lambda} \to 0 \text{ as } t \to +\infty\},$$
$$\mathfrak{R}(\Lambda) = \{x \in M : |X(t,x;0)|_{\Lambda} \to 0 \text{ as } t \to -\infty\}.$$

Define a relation on $\mathcal{W} \subset M$ and $\mathcal{D} \subset M$ by $\mathcal{W} \prec \mathcal{D}$ if $\mathfrak{A}(\mathcal{W}) \cap \mathfrak{R}(\mathcal{D}) \neq \oslash$.

Definition 2. [30] Let $\Lambda_1, \ldots, \Lambda_k$ be a decomposition of Λ, then

1. An r-cycle ($r \geq 2$) is an ordered r-tuple of distinct indices $i_1, \ldots, i_r$ such that $\Lambda_{i_1} \prec \ldots \prec \Lambda_{i_r} \prec \Lambda_{i_1}$.
2. A 1-cycle is an index i such that $\mathfrak{R}(\Lambda_i) \cap \mathfrak{A}(\Lambda_i) \setminus \Lambda_i \neq \oslash$.
3. A filtration ordering is a numbering of the Λ_i so that $\Lambda_i \prec \Lambda_j \Rightarrow i \leq j$.

As we can conclude from Definition 2, existence of an r-cycle with $r \geq 2$ is equivalent to existence of a heteroclinic orbit[2] for (2). Furthermore, existence of a 1-cycle implies existence of a homoclinic orbit for (2).

Definition 3. The set $\mathcal{W}$ is called decomposable if it admits a finite decomposition without cycles, $\mathcal{W} = \bigcup_{i=1}^{k} \mathcal{W}_i$, for some non-empty disjoint compact sets $\mathcal{W}_i$, which form a filtration ordering of $\mathcal{W}$, as detailed in definitions 1 and 2.

B. Robustness notions

The following robustness notions for systems represented by (1) have been introduced in [3], [4] (see also [11] for a survey on the ISS framework).

Definition 4. We say that the system (1) has the practical asymptotic gain (pAG) property with respect to $\mathcal{W}$ if there exist $\eta \in \mathcal{K}_\infty$ and $q \in \mathbb{R}_+$ such that for all $x \in M$ and all measurable essentially bounded inputs $d(\cdot)$ the solutions are defined for all $t \geq 0$ and the following holds:

$$\limsup_{t \to +\infty} |X(t,x;d)|_{\mathcal{W}} \leq \eta(\|d\|_\infty) + q.$$

If $q = 0$, then we say that the asymptotic gain (AG) property holds.

Definition 5. We say that the system (1) has the limit property (LIM) with respect to $\mathcal{W}$ if there exists $\mu \in \mathcal{K}_\infty$ such that for all $x \in M$ and all measurable essentially bounded inputs $d(\cdot)$ the solutions are defined for all $t \geq 0$ and the following holds:

$$\inf_{t \geq 0} |X(t,x;d)|_{\mathcal{W}} \leq \mu(\|d\|_\infty).$$

Definition 6. We say that the system (1) has the practical global stability (pGS) property with respect to $\mathcal{W}$ if there exist $\beta \in \mathcal{K}_\infty$ and $q \in \mathbb{R}_+$ such that for all $x \in M$ and measurable essentially bounded inputs $d(\cdot)$ the following holds for all $t \geq 0$:

$$|X(t,x;d)|_{\mathcal{W}} \leq q + \beta(\max\{|x|_{\mathcal{W}}, \|d\|_\infty\}).$$

It has been shown in [3], [4] that to characterize pAG property in terms of Lyapunov functions the following notion is appropriate.

Definition 7. We say that a $\mathcal{C}^1$ function $V : M \to \mathbb{R}_+$ is a practical ISS-Lyapunov function for (1) if there exists $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha_3$ and γ, and scalars $q \geq 0$ and $c \geq 0$ such that

$$\alpha_1(|x|_{\mathcal{W}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{W}} + c),$$

the function V is constant on each $\mathcal{W}_i$ and the following dissipation holds:

$$DV(x)f(x,d) \leq -\alpha_3(|x|_{\mathcal{W}}) + \gamma(|d|) + q.$$

If the latter inequality holds for $q = 0$, then V is said to be an ISS-Lyapunov function.

[2]The union of fixed points and the trajectories connecting them (heteroclinic orbits connect distinct points, and homoclinic orbits relate a point to itself) [21].

Note that existence of α_2 and c follows (without any additional assumptions) by standard continuity arguments.

The main result of [3], [4] relating these robust stability properties is stated below, it extends the results of [37], [39] obtained for connected sets.

Theorem 1. *Consider a nonlinear system as in (1) and let a compact invariant set containing all α- and ω-limit sets of (2) $\mathcal{W}$ be decomposable (in the sense of Definition 3). Then the following facts are equivalent:*

1. *The system admits an ISS-Lyapunov function;*
2. *The system enjoys the AG property;*
3. *The system admits a practical ISS-Lyapunov function;*
4. *The system enjoys the pAG property;*
5. *The system enjoys the LIM and the pGS properties.*

Definition 8. [4] Suppose that a nonlinear system as in (1) satisfies the assumptions and the list of equivalent properties of Theorem 1, then this system is called ISS with respect to the set $\mathcal{W}$.

C. Boundedness of solutions of periodic systems

As outlined in Section I, the present paper is dedicated to the stability analysis of periodic systems [20], [46]. More precisely, we assume in the following that for the system (1) there exists $\xi \in M$, $\xi \neq 0$, such that for all $x \in M$

$$f(x,0) = f(x+\xi,0).$$

Roughly speaking, in such a case there exists a coordinate transformation such that $M = \mathbb{R}^k \times \mathbb{S}^q$, where $n = k+q$ and $\mathbb{S}$ is the unit sphere.

Next, we recall a sufficient criterion derived in [27], [20], [46], which allows to establish *boundedness* of solutions of periodic systems. To this end consider a special case of the system (2) given by

$$f(x,0) = Px + b\varphi(c^T x),$$

with $M = \mathbb{R}^n$, where $P \in \mathbb{R}^{n\times n}$ is a singular matrix, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^n$, $\varphi : \mathbb{R} \rightrightarrows \mathbb{R}$ is a Δ-periodical set-valued function, which is upper semicontinuous, with a nonempty, convex and closed set of values for any value of its argument. We note that a time-varying version of φ has been considered in [20], [46], but we restrict ourselves to the autonomous version of φ. Then under these restrictions and for any initial condition $x_0 \in \mathbb{R}^n$ the system (2) has a solution $X(t, x_0; 0)$. Assume also that for all $\sigma \in \mathbb{R} \setminus \{0\}$ and all $\phi \in \varphi(\sigma)$

$$\mu_1 \leq \frac{\phi}{\sigma} \leq \mu_2;\ \mu_1^{-1}\mu_2^{-1}\varphi(0) = 0$$

for some $\mu_1 \in \mathbb{R} \cup \{-\infty\}$ and $\mu_2 \in \mathbb{R} \cup \{+\infty\}$. The periodicity of φ implies that either $\mu_1 < 0$, $\mu_2 > 0$ or $\mu_1 = \mu_2 = 0$, and the latter case is excluded from consideration due to its triviality.

Theorem 2. *[27], [20], [46] Assume that there exists $\lambda > 0$ such that:*

1) the matrix $P + \lambda I_n$, where $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix, has $n-1$ eigenvalues with negative real parts;

2) for all $\omega \in \mathbb{R}$

$$\mu_1^{-1}\mu_2^{-1} + (\mu_1^{-1} + \mu_2^{-1})Re\chi(i\omega - \lambda) + |\chi(i\omega - \lambda)|^2 \leq 0,$$

where $\chi(s) = c^T(P - sI_n)^{-1}b$.

Then, for any initial condition $x_0 \in \mathbb{R}^n$ the solution $X(t, x_0; 0)$ of the periodic system (2) is bounded for $t \in [0, +\infty)$.

The proof of this theorem (see Theorem 4.3.1 in [20], or Theorem 4.7 in [46]) is based on the fact that under the introduced conditions there is $H = H^T \in \mathbb{R}^{n\times n}$ (which has one negative and $n-1$ positive eigenvalues) such that for $V_0(x) = x^T H x$ we have that $dV_0(x(t))/dt \leq -2\lambda V_0(x(t))$ for all $t \in [0, +\infty)$, which implies that the set $\Omega_0 = \{x \in \mathbb{R}^n : V_0(x) \leq 0\}$ is invariant for (2), *i.e.* $X(t, x_0; 0) \in \Omega_0$ for all $t \in [0, +\infty)$ provided that $x_0 \in \Omega_0$. Next, introducing the functions $V_j(x) = V_0(x - j\delta)$ and sets $\Omega_j = \{x \in \mathbb{R}^n : V_j(x) < 0\}$, where j is any integer and the vector $\delta \in \mathbb{R}^n$ satisfies the conditions $\delta \neq 0$, $P\delta = 0$

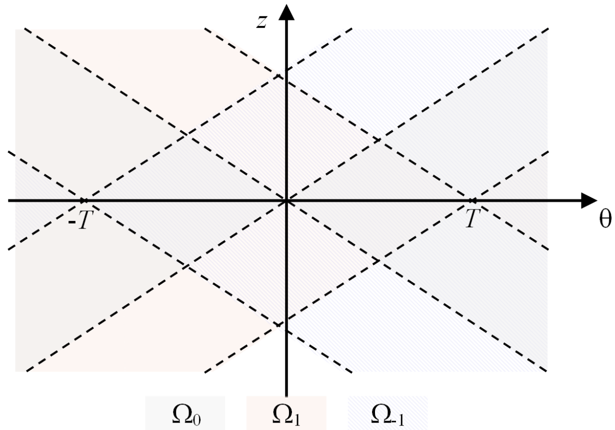

Figure 1. Illustration of the sets Ω_j for $j \in \{-1, 0, 1\}$

and $c^T\delta = \Delta$, by periodicity of f in (2) we obtain that $dV_j(x(t))/dt \leq -2\lambda V_j(x(t))$ for all $t \in [0, +\infty)$, then the sets Ω_j are invariant for (2). Finally, it is shown in [20], [46] that for any $x_0 \in \mathbb{R}^n$ there is an index j_0 such that $x_0 \in \Gamma_{j_0}$, where $\Gamma_j = \Omega_j \cap \Omega_{-j} \cap \{x \in \mathbb{R}^n : |h^T x| \leq j|h^T\delta|\}$ with $h \in \mathbb{R}^n$ being the eigenvector of the matrix H corresponding to the negative eigenvalue. As it has been shown above $X(t, x_0; 0) \in \Gamma_{j_0}$ for all $t \in [0, +\infty)$ (since it is true for $\Omega_{j_0} \cap \Omega_{-j_0}$). In addition the set Γ_{j_0} is bounded, which was necessary to prove. In other words, an important observation of [27], [20], [46] is that any intersection of the sets Ω_j for all integers j forms a kind of cell cover of $\mathbb{R}^n$, where each cell is bounded and invariant. Therefore, this framework is commonly known as cell structure approach.

The following example illustrates the main idea of Theorem 2.

Example 1. Suppose (2) is given by $x = [z, \theta]^\top \in \mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$. Furthermore, suppose f is periodic in θ with period $T > 0$, *i.e.*, for any integer $j = \pm 1, \pm 2, \ldots$, we have that $f(z, \theta + Tj) = f(z, \theta)$. Following Theorem 2, suppose that there exists a function $V : \mathbb{R}^2 \to \mathbb{R}$, such that

$$V(0,0) = 0, \quad V(z, 0) > 0 \quad \forall z \in \mathbb{R} \setminus \{0\},$$
$$\dot{V}(x) \leq -\lambda V(x) \quad \forall x \in \mathbb{R}^2.$$

By periodicity of f in T for any integer j

$$\dot{V}(z, \theta - Tj) \leq -\lambda V(z, \theta - Tj) \quad \forall (z, \theta) \in \mathbb{R}^2.$$

Denote $V_j(x) = V(z, \theta - Tj)$, then the invariant sets $\Omega_j = \{x \in \mathbb{R}^2 : V_j(x) < 0\}$ are employed in Theorem 2 to establish boundedness of solutions for the system and they are illustrated in Fig. 1. In general, the boundary at which V_j changes sign is not a straight line, but can be of arbitrary shape. The straight lines are chosen here to ease the illustration of the main idea and also to motivate the notion of cell structures.

III. Problem statement

The main contribution of the present work is the derivation of necessary and sufficient conditions under which a periodic system possesses the ISS properties given in Definition 8. This is achieved by combining the cell structure approach presented in the proof of Theorem 2 (and firstly introduced in [27]) with the ISS approach for multistable systems of [3], [4]. The fundamental difference between the theories given in subsections II-B and II-C is that the former performs an analysis on a manifold M, while the latter considers a multistable system in $\mathbb{R}^n$. To this end, let $M = \mathbb{R}^k \times \mathbb{S}^q$ (where $\mathbb{S}$ is the unit sphere) with $n = k + q$ and denote $x = (z, \theta) \in M$ with $z \in \mathbb{R}^k$ and $\theta \in \mathbb{S}^q$, then by embedding (2) in $\mathbb{R}^n$ and due to continuity of f we obtain that for all $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n$

$$f(\tilde{x}, 0) = f(\tilde{x} + \xi, 0), \ \xi = [\underbrace{0, \ldots, 0}_{k}, \underbrace{2\pi, \ldots 2\pi}_{q}] \in \mathbb{R}^n.$$

In this case for any $\tilde{x}_0 \in \mathbb{R}^n$ there is a unique and, at least, locally in time defined solution of the system (1) $\tilde{X}(t, \tilde{x}_0; d) \in \mathbb{R}^n$. Denote

$$\mathcal{P} : \mathbb{R}^n \to M$$

as the projection from $\mathbb{R}^n$ to M (that is just a modulus of the last q coordinates over 2π). Obviously, for any $\tilde{x}_0 \in \mathbb{R}^n$ the solution $\tilde{X}(t, \tilde{x}_0; d) \in \mathbb{R}^n$ of (1) can be projected to the solution $X(t, x_0; d) \in M$ with $x_0 = \mathcal{P}(\tilde{x}_0) \in M$, then both solutions are defined on the same time interval and $X(t, x_0; d) = \mathcal{P}(\tilde{X}(t, \tilde{x}_0; d))$ for all such instants of time. Similarly, the set $\mathcal{W} \subset M$, containing all α- and ω-limit sets of (2), can be extended to the whole $\mathbb{R}^n$ by using the periodicity of the last q variables, which we will denote by $\tilde{\mathcal{W}}$. Note that the set $\tilde{\mathcal{W}}$ becomes unbounded in $\mathbb{R}^n$, in a common case. Then $|\tilde{x}|_{\tilde{\mathcal{W}}} = \inf_{y \in \tilde{\mathcal{W}}} |\tilde{x} - y|$ is a distance to the set $\tilde{\mathcal{W}}$ for $\tilde{x} \in \mathbb{R}^n$.

Recall that the ISS-Lyapunov function introduced in Definition 7 should be positive definite with respect to the distance to the set $\mathcal{W}$, while the functions proposed in [27] for the analysis of boundedness of trajectories of the periodic system (2) are sign-indefinite. Usually sign-indefinite functions with a sign-definite derivative are used to establish instability of (2), *e.g.* Chetaev functions [10], [16]. Yet, clearly, the definiteness relaxation of a Lyapunov function to show stability might simplify significantly its construction. As demonstrated in [27] this applies in particular to periodic systems. Therefore, inspired by [27], we introduce the following characterization of the ISS property with respect to the set $\mathcal{W}$ for a periodic system:

Definition 9. We say that a $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ is a practical ISS-Leonov function for (1) with $M = \mathbb{R}^k \times \mathbb{S}^q$ if there exist functions $\alpha_1, \alpha_2, \sigma, \gamma \in \mathcal{K}_\infty$, a continuous function $\lambda : \mathbb{R} \to \mathbb{R}$, $\lambda \in \mathcal{K}_\infty$ for nonnegative arguments, and scalars $r \geq 0$, $g \geq 0$ such that for all $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n$ and $d \in \mathbb{R}^m$

$$\alpha_1(|\tilde{x}|_{\tilde{\mathcal{W}}}) - \sigma(|\tilde{\theta}|) \leq V(\tilde{x}) \leq \alpha_2(|\tilde{x}|_{\tilde{\mathcal{W}}} + g), \tag{3}$$

and the following dissipation holds:

$$DV(\tilde{x}) f(\tilde{x}, d) \leq -\lambda(V(\tilde{x})) + \gamma(|d|) + r. \tag{4}$$

If the latter inequality holds for $r = 0$, then V is said to be an ISS-Leonov function.

Let us stress that an ISS-Leonov function V can be continuously differentiable on $\mathbb{R}^n$, but discontinuous on M, while an ISS-Lyapunov function should be continuously differentiable on M (*i.e.* in this case V should be 2π-periodic in θ), which is another relaxation in Definition 9 compared to Definition 7. Therefore, any ISS-Lyapunov function is a practical ISS-Leonov function for a periodic system (1) since for any $\tilde{x} \in \mathbb{R}^n$ and any $\sigma \in \mathcal{K}_\infty$:

$$\begin{aligned} \alpha_1(|\tilde{x}|_{\mathcal{W}}) - \sigma(|\tilde{\theta}|) &\leq \alpha_1(|\tilde{x}|_{\mathcal{W}}), \\ -\alpha_3(|\tilde{x}|_{\mathcal{W}}) &\leq -\alpha_3(0.5[|\tilde{x}|_{\mathcal{W}} + c]) + \alpha_3(c) \\ &\leq -\alpha_3(0.5\alpha_2^{-1}(V(\tilde{x}))) + \alpha_3(c). \end{aligned}$$

Remark 1. If $0 \in \mathcal{W}$, then without loosing generality the property (3) in Definition 9 can be replaced by the following one:

$$\alpha_1(|\tilde{z}|) - \sigma(|\tilde{\theta}|) \leq V(\tilde{x}) \leq \alpha_2(|\tilde{x}|_{\tilde{\mathcal{W}}} + g). \tag{5}$$

In the remainder of this work, it is shown that the existence of a practical ISS-Leonov function is an equivalent characterization of the ISS property from Definition 8 for a periodic system (1).

IV. Main result

If $V : M \to \mathbb{R}$ is a continuously differentiable function admitting relations (3) for all $x \in M$ and some $\alpha_1, \alpha_2, \sigma \in \mathcal{K}_\infty$, then by adding a constant $w > 0$ the new function $V(x) + w$ can be made positive definite. Therefore, the introduction of V as a function from $\mathbb{R}^n$ to $\mathbb{R}$ is fundamental in Definition 9.

Lemma 1. *Let $M = \mathbb{R}^k \times \mathbb{S}^q$ with $n = k + q$, and $\mathcal{W} \subset M$ be a compact invariant set. Then existence of a practical ISS-Leonov function for (1) implies pGS and pAG properties with respect to $\mathcal{W}$.*

Proof. In this case by Definition 9 there are functions $\alpha_1, \alpha_2, \sigma, \gamma, \lambda \in \mathcal{K}_\infty$ and scalars $r \geq 0$, $g \geq 0$ such that for all $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n$ and $d \in \mathbb{R}^m$ the relations (3) and the inequality (4) are satisfied for a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$. Take $\tilde{x}_0 \in \mathbb{R}^n$ and consider a solution $\tilde{X}(t, \tilde{x}_0; d)$ of the system (1), which is defined at least locally in time for $t \in [0, T)$ for some $T > 0$. Then we can introduce two subsets of instants of time

$$\begin{aligned} \mathcal{T}_+ &= \{t \in [0, T) : V(\tilde{X}(t, \tilde{x}_0; d)) \geq 0\}, \\ \mathcal{T}_- &= \{t \in [0, T) : V(\tilde{X}(t, \tilde{x}_0; d)) < 0\}, \end{aligned}$$

such that $[0, T) = \mathcal{T}_+ \cup \mathcal{T}_-$. For any $t \in \mathcal{T}_-$ we have from (3) (for brevity we will use below the notation $\tilde{x}(t) = (\tilde{z}(t), \tilde{\theta}(t)) = \tilde{X}(t, \tilde{x}_0; d)$):

$$\alpha_1(|\tilde{x}(t)|_{\tilde{\mathcal{W}}}) - \sigma(|\tilde{\theta}(t)|) \leq V(\tilde{x}(t)) < 0,$$

then

$$\alpha_1(|\tilde{x}(t)|_{\tilde{\mathcal{W}}}) < \sigma(|\tilde{\theta}(t)|).$$

For any $t \in \mathcal{T}_+$, if such a t is an isolated element of $\mathcal{T}_+$ then as previously we get

$$\alpha_1(|\tilde{x}(t)|_{\tilde{\mathcal{W}}}) \leq \sigma(|\tilde{\theta}(t)|).$$

If $t \in \mathcal{T}_+$ is not an isolated element of $\mathcal{T}_+$, then there is an interval $[t_s, t_e] \subset \mathcal{T}_+$ and $t \in [t_s, t_e]$ with $V(\tilde{x}(t)) \geq 0$ for all such instants of time. Therefore, on the interval $[t_s, t_e]$ the standard results from ISS theory [37], [39] can be applied, and there exists a function $\beta \in \mathcal{KL}$ such that for all $t \in [t_s, t_e]$:

$$V(\tilde{x}(t)) \leq \beta(V(\tilde{x}(t_s)), t - t_s) + \lambda^{-1}(\gamma(\|d\|_\infty) + r).$$

Next, similarly from (3) we obtain that

$$\begin{aligned} \alpha_1(|\tilde{x}(t)|_{\tilde{\mathcal{W}}}) \leq\ & \sigma(|\tilde{\theta}(t)|) + \beta(V(\tilde{x}(t_s)), t - t_s) \\ & + \lambda^{-1}(\gamma(\|d\|_\infty) + r) \end{aligned}$$

for all $t \in [t_s, t_e]$ (the inverse of λ is well defined for nonnegative arguments). For t_s two options are possible: either $t_s = 0$ and $V(\tilde{x}(t_s)) \in \mathbb{R}_+$ or $t_s > 0$ and $V(\tilde{x}(t_s)) = 0$. Hence, combining the above estimates we obtain for any $t \in [0, T)$:

$$\alpha_1(|\tilde{x}(t)|_{\tilde{\mathcal{W}}}) \leq \sigma(|\tilde{\theta}(t)|) + \beta(V(\tilde{x}_0), t) + \lambda^{-1}(\gamma(\|d\|_\infty) + r).$$

Recall that q is the dimension of the periodic state variable. Denote by $j = [j_1, \ldots, j_q]$ a multi-index vector, where j_s is an integer for all $s = 1, \ldots, q$, and let us consider the behavior of the functions $V_j(x) = V((z, \theta - 2\pi j))$ for any such vector j (then $V_0(x) = V(x)$). From (3) we have for all $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n$:

$$\alpha_1(|\tilde{x}|_{\tilde{\mathcal{W}}}) - \sigma(|\tilde{\theta} - 2\pi j|) \leq V_j(\tilde{x}) \leq \alpha_2(|\tilde{x}|_{\tilde{\mathcal{W}}} + g),$$

since $|\tilde{x}|_{\tilde{\mathcal{W}}} = |(\tilde{z}, \tilde{\theta})|_{\tilde{\mathcal{W}}} = |(\tilde{z}, \tilde{\theta} - 2\pi j)|_{\tilde{\mathcal{W}}}$ (with $\tilde{\mathcal{W}}$ being extended from $\mathcal{W} \subset M$ to $\mathbb{R}^n$ by using the periodicity in θ). As for a $\mathcal{C}^1$ periodic system (1) $f(\tilde{x}, d) = f((\tilde{z}, \tilde{\theta}), d) = f((\tilde{z}, \tilde{\theta} - 2\pi j), d)$, from (4) for all $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n$ and $d \in \mathbb{R}^m$ we obtain:

$$DV_j(\tilde{x}) f(\tilde{x}, d) \leq -\lambda(V_j(\tilde{x})) + \gamma(|d|) + r.$$

Then repeating the same calculations as above for the function V, for any multi-index vector j the following estimate is justified for all $t \in [0, T)$:

$$\begin{aligned} \alpha_1(|\tilde{x}(t)|_{\tilde{\mathcal{W}}}) &\leq \sigma(|\tilde{\theta}(t) - 2\pi j|) + \beta(V_j(\tilde{x}_0), t) \\ &\quad + \lambda^{-1}(\gamma(\|d\|_\infty) + r) \\ &\leq \sigma(|\tilde{\theta}(t) - 2\pi j|) + \beta(\alpha_2(|\tilde{x}_0|_{\tilde{\mathcal{W}}} + g), t) \\ &\quad + \lambda^{-1}(\gamma(\|d\|_\infty) + r). \end{aligned}$$

Since this inequality is satisfied for all possible j, then the minimum with respect to j of the right-hand side can be calculated. Note that the right-hand side is composed of three independent nonnegative terms that can be minimized independently and that only one of them is dependent on j. Furthermore, by construction

$$\min_j |\tilde{\theta}(t) - 2\pi j| \leq \sqrt{q}\pi.$$

Thus, for all $t \in [0, T)$

$$\begin{aligned} \alpha_1(|\tilde{x}(t)|_{\tilde{\mathcal{W}}}) \quad &\leq \quad \sigma(\sqrt{q}\pi) + \beta(\alpha_2(|\tilde{x}_0|_{\tilde{\mathcal{W}}} + g), t) \\ &\quad\quad + \lambda^{-1}(\gamma(\|d\|_\infty) + r) \end{aligned}$$

and the distance to the set $\tilde{\mathcal{W}}$ is bounded considering the trajectory $\tilde{X}(t, \tilde{x}_0; d)$ in $\mathbb{R}^n$. By projecting it back into M, *i.e.* considering $X(t, x_0; d) = \mathcal{P}(\tilde{X}(t, \tilde{x}_0; d))$ for all $t \in [0, T)$ with $x_0 = \mathcal{P}(\tilde{x}_0)$, we obtain that $|X(t, x_0; d)|_{\mathcal{W}} = |\tilde{X}(t, \tilde{x}_0; d)|_{\tilde{\mathcal{W}}}$ is also bounded in M. Since for any $x \in M$ there is a constant $h \geq 0$ such that $|x| \leq |x|_{\mathcal{W}} + h$, then the state trajectory $X(t, x_0; d)$ is bounded in M, which implies that $T = +\infty$. Moreover, the system (1) has pGS property with respect to the set $\mathcal{W}$ from Definition 6:

$$|X(t, x_0; d)|_{\mathcal{W}} \leq r' + \beta'(\max\{|x_0|_{\mathcal{W}}, \|d\|_\infty\}),$$

where

$$r' = \alpha_1^{-1}(4\sigma(\sqrt{q}\pi)) + \alpha_1^{-1}(4\beta(\alpha_2(2g), 0)) + \alpha_1^{-1}(4\lambda^{-1}(2r)),$$
$$\beta'(s) = \max\{\alpha_1^{-1}(4\beta(\alpha_2(2s), 0)), \alpha_1^{-1}(4\lambda^{-1}(2\gamma(s)))\}.$$

Finally,

$$\limsup_{t \to +\infty} |X(t, x_0; d)|_{\mathcal{W}} \leq \eta(\|d\|_\infty) + r'',$$

where $\eta(s) = \alpha_1^{-1}(2\lambda^{-1}(2\gamma(s)))$ and $r'' = \alpha_1^{-1}(2\sigma(\sqrt{q}\pi) + \lambda^{-1}(2r))$, and the system (1) also admits pAG property from Definition 4. This completes the proof. $\square$

We are now in the position to present the main result of this work.

Theorem 3. *Let $M = \mathbb{R}^k \times \mathbb{S}^q$ with $n = k + q$, and a compact invariant set containing all α- and ω-limit sets of (2) $\mathcal{W} \subset M$ be decomposable (in the sense of Definition 3). Then, for (1) the following properties are equivalent:*

(a) *ISS with respect to the set $\mathcal{W}$;*

(b) *there is a practical ISS-Leonov function.*

Proof. By Theorem 1, if (1) possesses ISS property with respect to the set $\mathcal{W}$, then there exists an ISS-Lyapunov function $V : M \to \mathbb{R}_+$ from Definition 7, which by Definition 9 is also a practical ISS-Leonov function. Thus, $(a) \Rightarrow (b)$, and it remains to prove that $(b) \Rightarrow (a)$. In such a case, by Lemma 1, the system admits pGS and pAG properties with respect to the set $\mathcal{W}$, which by Theorem 1 imply the ISS property with respect to the set $\mathcal{W}$. $\square$

Note that from this result in general, an ISS-Leonov function is only sufficient for the ISS property of (1):

Corollary 1. *Let $M = \mathbb{R}^k \times \mathbb{S}^q$ with $n = k + q$, and a compact invariant set containing all α- and ω-limit sets of (2) $\mathcal{W}$ be decomposable (in the sense of Definition 3). Then for (1) existence of an ISS-Leonov function implies the ISS property with respect to the set $\mathcal{W}$.*

Proof. Clearly, an ISS-Leonov function is also a practical ISS-Leonov function, which by Theorem 3 implies the result. $\square$

The practical interest of the proposed theory is illustrated via a benchmark example [19] in the next section.

V. Application to a nonlinear pendulum

We demonstrate the usefulness of the proposed approach by performing a global ISS analysis of a *forced* (*i.e.* with a biased external input) nonlinear pendulum:

$$\begin{aligned} \dot{\theta}(t) &= z(t), \\ \dot{z}(t) &= -\kappa z(t) - \omega^2 \sin(\theta(t)) + c + d(t), \end{aligned} \tag{6}$$

where $\theta(t) \in \mathbb{S}$ and $z(t) \in \mathbb{R}$ are the angular position and the angular velocity of the pendulum, $x = (z, \theta) \in M = \mathbb{R} \times \mathbb{S}$, $\kappa > 0$ and $\omega > 0$ are two parameters, $c \in \mathbb{R}$ is the input bias, and $d(t) \in \mathbb{R}$ is an external disturbance (a locally essentially bounded and measurable signal).

In addition to being a well-known academic example, the forced pendulum dynamics also is frequently encountered in many engineering applications. For example, reduced-order synchronous generators or droop-controlled power converters can be represented as forced pendula [15], [17], [36]. However, even for this simplified setup available results in the literature are either limited to *local* statements [1] or, if global, restricted to the *unforced* dynamics (6), *i.e.*, with $c = d(t) = 0$, [45]. We remark that both limitations (local or $c = d(t) = 0$) arise from the difficulty of constructing a Lyapunov function for the forced dynamics (6) that is continuous in θ and *globally* positive definite with respect to the equilibria of (6). It is straightforward to see that for $d(t) = 0$ the latter are given by $[\theta_0, 0]$ and $[\pi - \theta_0, 0]$, where $\theta_0 = \operatorname{asin}(c\omega^{-2})$.

In the ISS context, the unperturbed version of the system (6), *i.e.* with $c = 0$, has been studied in [3], [4]. For $c = d(t) = 0$, (6) has two equilibria $[0; 0]$ and $[\pi; 0]$ (the former is attractive and the latter one is a saddle-point). Thus, for $c = 0$ the set $\mathcal{W} = \{[0, 0] \cup [\pi, 0]\}$ is compact and containing all α- and ω-limit sets of (6). In addition, it is straightforward to check that $\mathcal{W}$ is decomposable in the sense of Definition 3. For that case, the ISS property of (6) has been shown in [3], [4] with respect to the set $\mathcal{W}$, and an ISS-Lyapunov function for (6) has been proposed in [15], [17]. Using that result, for the case $|c| < \omega^2$, the global convergence to one of the two equilibria $[\theta_0, 0]$ or $[\pi - \theta_0, 0]$ has been proven in [17] under some restrictions on the values of the parameters c, κ, ω and by using an additional discontinuous Lyapunov function for a local analysis.

In the following, we demonstrate the ISS property of (6) under less restrictive conditions than in [17] by using the ISS-Leonov function framework proposed above. For this purpose, assume that $|c| < \omega^2$ and consider

$$V(x) = 0.5z^2 + \omega^2 w(\theta - \theta_0),$$
$$w(s) = \cos(\theta_0) - \cos(s + \theta_0) - \sin(\theta_0)s - u\cos(\theta_0)s^2,$$

where $u \in \mathbb{R}$ is a parameter to be defined later. Note that w is not periodic in θ, thus V cannot be an ISS-Lyapunov function, but it can be considered as an ISS-Leonov function candidate. Straightforward calculations yield:

$$\begin{aligned} w'(s) &= \sin(s + \theta_0) - \sin(\theta_0) - 2u\cos(\theta_0)s, \\ w''(s) &= \cos(s + \theta_0) - 2u\cos(\theta_0). \end{aligned}$$

Since $\cos(\theta_0) = \sqrt{1 - c^2\omega^{-4}} > 0$, then $w''(0) < 0$ for $u > 0.5$ and there exist $u^* > 0.5$ such that $w'(s) = 0$ only for $s = 0$ with $u \geq u^*$. Indeed, the equation $w''(s) = 0$ has no solution for a sufficiently high u and $w'(s)$ is strictly decreasing in such a case. Therefore, for $u \geq u^*$ these properties imply that $w(s) < 0$ for all $s \neq 0$ and $w(0) = 0$, then for $u > u^*$there exist $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that

$$-\epsilon_1 s^2 \leq w(s) \leq -\epsilon_2 s^2.$$

Hence, for all $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^2$:

$$0.5\tilde{z}^2 - \epsilon_1\omega^2(\tilde{\theta} - \theta_0)^2 \leq V(\tilde{x}) \leq 0.5\tilde{z}^2 - \epsilon_2\omega^2(\tilde{\theta} - \theta_0)^2$$

and the relations (3) are satisfied. Let us check (4):

$$\begin{aligned}
\dot{V} &= \tilde{z}d - \kappa\tilde{z}^2 - 2u\omega^2\cos(\theta_0)(\tilde{\theta}-\theta_0)\tilde{z} \\
&= \gamma d^2 - \lambda\left(\frac{\tilde{z}^2}{2} - \epsilon_2\omega^2(\tilde{\theta}-\theta_0)^2\right) \\
&\quad - \begin{bmatrix} \tilde{\theta}-\theta_0 \\ \tilde{z} \\ d \end{bmatrix}^T Q \begin{bmatrix} \tilde{\theta}-\theta_0 \\ \tilde{z} \\ d \end{bmatrix}, \\
Q &= \begin{bmatrix} \lambda\epsilon_2\omega^2 & u\omega^2\cos(\theta_0) & 0 \\ u\omega^2\cos(\theta_0) & \kappa-\frac{\lambda}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \gamma \end{bmatrix},
\end{aligned}$$

where $\lambda > 0$ and $\gamma > 0$ are two tuning parameters. By applying the Schur complement to Q we obtain that $Q \geq 0$ provided that

$$\gamma > 0,\ X \geq 0,$$
$$X = \begin{bmatrix} \lambda\epsilon_2\omega^2 & u\omega^2\cos(\theta_0) \\ u\omega^2\cos(\theta_0) & \kappa-\frac{\lambda}{2}-\frac{1}{4\gamma} \end{bmatrix}.$$

Performing straightforward calculations we obtain that $X \geq 0$ if and only if

$$\gamma > 0,\ \lambda > 0,$$
$$\kappa > \frac{\lambda}{2} + \frac{1}{4\gamma},\ \lambda\left(\kappa - \frac{\lambda}{2} - \frac{1}{4\gamma}\right) \geq \frac{u^2}{\epsilon_2}\omega^2\cos^2(\theta_0).$$

Resolving the last inequality with respect to $\lambda > 0$ we obtain that all conditions above are satisfied if

$$\kappa > \sqrt{\frac{2(\omega^4-c^2)}{\epsilon_2}}\frac{u}{\omega}. \tag{7}$$

Thus, if (7) is true, then there exists $\lambda > 0$ and $\gamma > 0$ such that $Q \geq 0$, and finally we obtain:

$$\begin{aligned}
\dot{V} &\leq \gamma d^2 - \lambda\left(\frac{\tilde{z}^2}{2} - \epsilon_2\omega^2(\tilde{\theta}-\theta_0)^2\right) \\
&\leq \gamma d^2 - \lambda V.
\end{aligned}$$

Consequently, V is an ISS-Leonov function for (6). By taking $d = 0$ it is easy to prove [17] that all solutions are bounded in that case and converge to one of the equilibria: $[\theta_0, 0]$ or $[\pi - \theta_0, 0]$. Then under the restriction on parameters (7) (ϵ_2 and u are also some functions of c, κ, ω), $\mathcal{W} = \{[\theta_0, 0] \cup [\pi - \theta_0, 0]\}$ is a compact set containing all α- and ω-limit sets of (6) for $d = 0$, and it is decomposable in the sense of Definition 3. Finally, by Theorem 3 the system (6) is ISS with respect to $\mathcal{W}$ under the restriction (7).

Remark 2. A slightly more general problem is the analysis of the system

$$\begin{aligned}
\dot{\phi}(t) &= z(t), \\
\dot{z}(t) &= -\kappa z(t) - a\sin(\phi(t)) - b\cos(\phi(t)) + c + d(t),
\end{aligned}$$

where $\phi(t) \in \mathbb{S}$ and $a \in \mathbb{R}$, $b \in \mathbb{R}$ are parameters. Yet, the above system can be reduced to (6) by defining:

$$\omega^2 = \sqrt{a^2+b^2},\ \theta(t) = \phi(t) + \varphi,\ \varphi = \arctan(b/a).$$

Remark 3. The values of u (or u^*) and ϵ_2 can be calculated numerically (finding an analytical solution is complicated since

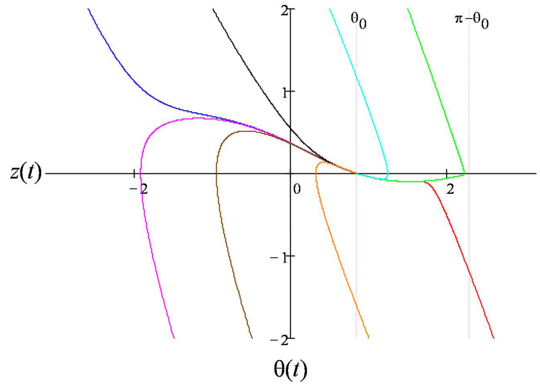

Figure 2. Simulation results for the system (6) with $d(t) = 0$ and for several arbitrarily chosen initial conditions

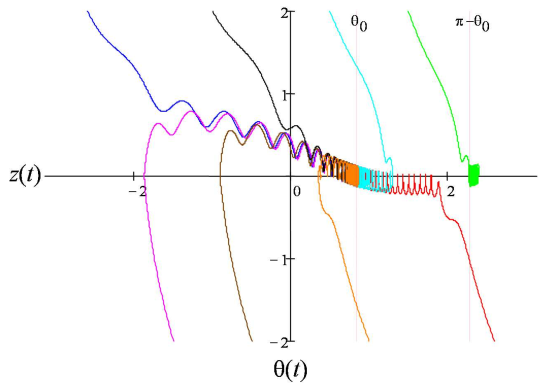

Figure 3. Simulation results for the system (6) with $d(t) = 1.1\sin(8.635t)$ and for several arbitrarily chosen initial conditions

$w'(s) = 0$ is a transcendent equation). Alternatively, the following (maybe conservative) estimates can be proposed:

$$\hat{u} = \frac{1}{\cos(\theta_0)},\ \hat{\epsilon}_2 = \frac{1}{4\cos(\theta_0)},$$

which follow from the observation that for $u \geq \frac{1}{2\cos(\theta_0)}$ the linear term $2u\cos(\theta_0)s$ is growing faster than $\sin(s+\theta_0) - \sin(\theta_0)$ in $w'(s)$ (note that $w'(0) = 0$). Then the restriction (7) takes the form:

$$\kappa > 2\sqrt{2}\sqrt[4]{\omega^4 - c^2}. \tag{8}$$

Example 2. Select $c = 0.75$ and $\omega = 1$, then $u^* \cong 0.7$ and $\epsilon_2 = \frac{1}{50}$ can be found numerically, and for any κ such that the restriction (8) is satisfied, that is $\kappa > 2.3$, the system (6) is ISS with respect to $\mathcal{W} = \{[0.848, 0] \cup [2.294, 0]\}$. Examples of the system trajectories with $\kappa = 2.5$ and $d = 0$ are given in Fig. 2, and for $d(t) = 1.1\sin(8.635t)$ in Fig. 3. Note that in Fig. 3 the unstable equilibrium captures one of the trajectories, which means that the applied input acts as a stabilizing feedforward control in this case. Clearly, the simulations confirm the conclusions of the proposed theory.

VI. Conclusions

We have derived necessary and sufficient conditions to establish the ISS property for multistable periodic systems, *i.e.*, systems the dynamics of which is periodic with respect to a part of the state variables. To prove this result and by building upon pioneering ideas in [27], we have introduced the concept of an ISS-Leonov function. Such a function is in general sign-indefinite and not continuously differentiable on the manifold where the system dynamics evolves. These achievements represent significant relaxations compared to the usual requirements on a standard ISS-Lyapunov function [3], [4]. The proposed approach is illustrated by providing a global analysis of a nonlinear pendulum with constant input. We expect the derived methodology to be applicable to many challenging engineering problems. Its application to power systems and microgrids is currently under investigation.

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